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# Blow up of solutions for a class of fourth order nonlinear pseudo-parabolic equation with a nonlocal source

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**Abstract**

In this paper, we consider the initial boundary value problem for a fourth order nonlinear pseudo-parabolic equation with a nonlocal source. By using the concavity method, we establish a blow-up result of the solutions under suitable assumptions on the initial energy.

**MSC:** 35B44; 35K30; 35K59

**Keywords:** blow up; fourth order; nonlinear pseudo-parabolic; nonlocal source; concavity method

## 1 Introduction

In this article, we are concerned with the following initial boundary value problem:

$$\begin{cases} u_t - \triangle u - \triangle u_t + \triangle^2 u = u^p(x,t)\int_\Omega K(x,y)u^{p+1}(y,t)\,dy, & x\in\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u=\frac{\partial u}{\partial \nu}=0 \text{ or } u=\triangle u=0, & x\in\partial\Omega, t>0,\end{cases} \tag{1.1}$$

where $p>0$, and $\Omega$ is a bounded domain of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$. Here, $\nu$ is the unit outward normal to $\partial\Omega$, and $K(x,y)$ is an integrable, real valued function such that $K(x,y)=K(y,x)$. It is well known that this type of equations describes a variety of important physical processes, such as the analysis of heat conduction in materials with memory, viscous flow in materials with memory [1], the theory of heat and mass exchange in stably stratified turbulent shear flow [2], the non-equilibrium water-oil displacement in porous strata [3], the aggregation of populations [4–6], the velocity evolution of ion-acoustic waves in a collisionless plasma when ion viscosity is invoked [7], filtration theory [8, 9], cell growth theory [10, 11], and so on. In population dynamics theory, the nonlocal term indicates that evolution of species at a point of space does not depend only on the nearby density but also on the total amount of species due to the effects of spatial inhomogeneity; see [4].

There have also been many profound results on the existence of global solutions and asymptotic behavior of the solutions for the initial boundary value problems and the initial value problems of fourth order nonlinear pseudo-parabolic equations.

In 1972, Kabanin [8] considered the following problem:

$$\begin{cases} u_t - \beta^2 u_{xxt} + \gamma u_{xxxx} = \alpha u_{xx}, \quad 0 < x < l, t > 0, \\ u(x,0) = \varphi(x), \quad 0 \le x \le l, \\ u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad 0 \le t \le T, \end{cases} \tag{1.2}$$

where $\alpha$, $\beta$, $\gamma$ are positive constants. A classical solution of this mixed problem is obtained through the Fourier method in the form of a series. Conditions sufficient for uniform convergence of this series are found.

In 1978, Bakiyevich and Shadrin [9] considered the following problem:

$$\begin{cases} u_t - \gamma u_{xxt} + \beta u_{xxxx} = \alpha u_{xx} + f(t,x), \quad x \in \mathbb{R}, t > 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $\alpha > 0$, $\beta \ge 0$, $\gamma > 0$ are constants. They showed that the solutions of this problem are expressed through the sum of convolutions of functions $\varphi(x)$ and $f(t,x)$ with corresponding fundamental solutions of the problem.

Zhao and Xuan [12] studied the following fourth order pseudo-parabolic equation:

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \quad x \in \mathbb{R}, t \ge 0. \tag{1.4}$$

They obtained the existence of the global smooth solutions for the initial value problem of (1.4) and discussed the convergence behavior of solutions as $\beta \to 0$.

Recently, Khudaverdiyev and Farhadova [13] discussed the following fourth order semilinear pseudo-parabolic equation:

$$u_t - \alpha u_{xxt} + u_{xxxx} = f(t,x,u,u_x,u_{xx},u_{xxx}), \quad 0 \le x \le 1, 0 \le t \le T, \tag{1.5}$$

with Ionkin type non-self-adjoint mixed boundary conditions, where $\alpha > 0$ is a fixed number. They proved the local existence for a generalized solution of the mixed problem under consideration by combining generalized contracted mapping principle and Schauder's fixed point principle and then proved the global existence for a generalized solution by means of Schauder's stronger fixed point principle.

The so-called viscous Cahn-Hilliard equation is also in a class of fourth order nonlinear pseudo-parabolic equations and can be considered as a special case of (1.5). In recent years, a lot of attention has been paid to the viscous Cahn-Hilliard equations. For more and deeper investigations of the stability analysis (as $t \to \infty$) and the asymptotic behavior of viscous Cahn-Hilliard models, we refer readers to [14, 15] and the references therein.

Since the study on blow-up solutions for nonlinear parabolic equation with nonlocal source by Levine in [16], many efforts have been made devoted to the study of blow-up properties for nonlocal semilinear parabolic equations. The upper bound and lower bound of the blow-up time, blow-up rate estimate, blow-up set, and blow-up profile of the blow-up solutions for a various of nonlocal semilinear parabolic equations with nonlocal source terms or nonlocal boundary condition have been widely studied in the last few decades; we refer the readers to [17–29] and the references cited therein.

Korpusov [30] considered a Sobolev type equation with a nonlocal source and obtained blow-up results under suitable conditions on initial data and nonlinear function. In [31], Bouziani studied the solvability of nonlinear pseudo-parabolic equation with a nonlocal boundary condition. More results on the global well-posedness for the nonlinear pseudo-parabolic equation with nonlocal source can be found in [1] and the references therein.

Motivated by the above-mentioned works, we investigate the blow-up behavior of solutions of the initial boundary value problem for a fourth order nonlinear pseudo-parabolic equation with a nonlocal source (1.1). By using the concavity method, we prove a finite time blow-up result under some assumption on the initial energy $E(0)$.

## 2 Preliminaries

In this section, we first state a local existence theorem, which can be obtained by Faedo-Galerkin methods. The interested readers are referred to Lions [32] or Escobedo and Herrero [33] for details.

**Theorem 2.1** *Assume that $p > 0$ and $u_0 \in H_0^2(\Omega)$. Then there exists a $T_m > 0$ for which problem* (1.1) *has a unique local solution $u \in C^1([0, T_m); H_0^2(\Omega))$ satisfying*

$$(u_t, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\triangle u, \triangle v) = \left(u^p(x,t) \int_\Omega K(x,y) u^{p+1}(y,t)\, dy, v\right), \tag{2.1}$$

*for all $v \in H_0^2(\Omega)$ and $t \in [0, T_m)$.*

Before stating our principal theorem, we note that the Fréchet derivative $f_u$ of the nonlinear function $f(u) = u^p(x,t) \int_\Omega K(x,y) u^{p+1}(y,t)\, dy$ is

$$\begin{aligned} f_u \cdot h(x,t) &= p u^{p-1}(x,t) h(x,t) \int_\Omega K(x,y) u^{p+1}(y,t)\, dy \\ &\quad + (p+1) u^p(x,t) \int_\Omega K(x,y) u^p(y,t) h(y,t)\, dy, \quad \forall u \in H^2(\Omega). \end{aligned}$$

Clearly $f_u$ is symmetric and bounded, so that the potential $F$ exists and is given by

$$\begin{aligned} F(u) &= \int_0^1 \big(f(\rho u), u\big)\, d\rho \\ &= \int_0^1 \int_\Omega \rho^p u^p(x,t) \left[\int_\Omega K(x,y) \rho^{p+1} u^{p+1}(y,t)\, dy\right] u(x,t)\, dx\, d\rho \\ &= \frac{1}{2p+2} \int_\Omega \int_\Omega K(x,y) u^{p+1}(x,t) u^{p+1}(y,t)\, dx\, dy. \end{aligned} \tag{2.2}$$

Now, differentiating the identity (2.2) with respect to $t$, it follows that

$$\begin{aligned} \frac{d}{dt} F(u) &= \frac{1}{2p+2} \frac{d}{dt} \int_\Omega \int_\Omega K(x,y) u^{p+1}(x,t) u^{p+1}(y,t)\, dx\, dy \\ &= \frac{1}{2} \int_\Omega \int_\Omega K(x,y) u^p(x,t) u^{p+1}(y,t) u_t(x,t)\, dx\, dy \\ &\quad + \frac{1}{2} \int_\Omega \int_\Omega K(x,y) u^p(y,t) u^{p+1}(x,t) u_t(y,t)\, dx\, dy \\ &= \int_\Omega \int_\Omega K(x,y) u^p(x,t) u^{p+1}(y,t) u_t(x,t)\, dx\, dy = \big(f(u), u_t\big), \end{aligned} \tag{2.3}$$

where we have used the symmetry of $K(x,y)$.

To obtain the blow-up result, we will introduce the energy function. We have

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\triangle u|^2\,dx - \frac{1}{2p+2}\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy. \tag{2.4}$$

**Lemma 2.1** *Let $p > 0$ and $u$ be a solution of the problem* (1.1). *Then $E(t)$ is non-increasing function, that is, $E'(t) \le 0$. Moreover, the following energy equality holds*:

$$E(t) + \int_0^t \left(|u_t|^2 + |\nabla u_t|^2\right)dx\,d\tau = E(0).$$

*Proof* Multiplying (1.1) by $u_t$ and integrating over $\Omega$, we have

$$\begin{aligned}&\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\triangle u|^2\,dx\\ &\quad= \int_\Omega u^p(x,t)\left[\int_\Omega K(x,y)u^{p+1}(y,t)\,dy\right]u_t(x,t)\,dx.\end{aligned}$$

Hence, from (2.3), we obtain

$$\begin{aligned}&\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\triangle u|^2\,dx\\ &\quad= \frac{1}{2p+2}\frac{d}{dt}\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy\end{aligned}$$

and

$$\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx + \frac{d}{dt}E(t) = 0. \tag{2.5}$$

Integrating (2.5) from 0 to $t$, we find

$$E(t) + \int_0^t\int_\Omega \left(|u_t|^2 + |\nabla u_t|^2\right)dx\,d\tau = E(0). \tag{2.6}$$

The proof of the Lemma 2.1 is completed. □

## 3 Blow up of solutions

Now, we will state the blow-up result of the solutions to the problem (1.1).

**Theorem 3.1** *Assume that $p > 0$ and $u_0 \in H_0^2(\Omega)$. If $u(x,t)$ is a solution of the problem* (1, 1) *and the initial data $u_0(x)$ satisfies*

$$\int_\Omega \left(|u_0|^2 + |\nabla u_0|^2\right)dx > \eta E(0), \tag{3.1}$$

*then the solution of problem* (1.1) *blows up in finite time*; *that is, the maximum existence time $T_{\max}$ of $u(x,t)$ is finite and*

$$\lim_{t\to T_{\max}^-}\int_0^t\int_\Omega \left(|u|^2 + |\nabla u|^2\right)dx\,d\tau = +\infty,$$

*where $\eta = \frac{\alpha}{m}$; $m = (\frac{\alpha}{2} - 1)\lambda_1$; $2 \leq \alpha \leq 2p + 2$; $\lambda_1$ is the first eigenvalue of operator $-\triangle$ under homogeneous Dirichlet boundary conditions.*

*Proof* The proof makes use of the so-called 'concavity method'. Multiplying (1.1) by $u$ and integrating over $\Omega$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\triangle u|^2\,dx \\
&\quad = \int_\Omega u^p(x,t)\left[\int_\Omega K(x,y)u^{p+1}(y,t)\,dy\right]u(x,t)\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\triangle u|^2\,dx \\
&\qquad - \int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy + \alpha E(u) - \frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx \\
&\qquad - \frac{\alpha}{2}\int_\Omega |\triangle u|^2\,dx + \frac{\alpha}{2p+2}\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \\
&\quad = \frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u|^2\,dx + \int_\Omega |\nabla u|^2\,dx\right] + \alpha E(u) \\
&\qquad + \left(\frac{\alpha}{2p+2} - 1\right)\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \\
&\qquad + \left(1 - \frac{\alpha}{2}\right)\int_\Omega |\nabla u|^2\,dx + \left(1 - \frac{\alpha}{2}\right)\int_\Omega |\triangle u|^2\,dx = 0.
\end{aligned} \tag{3.2}$$

We consider the following function:

$$H(t) = \int_\Omega \left(|u|^2\,dx + |\nabla u|^2\right)dx - \eta E(0). \tag{3.3}$$

From (3.2), (3.3), Lemma 2.1, and Poincaré's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}H(t) \\
&\quad = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u|^2 + |\nabla u|^2\right)dx \\
&\quad = \left(\frac{\alpha}{2} - 1\right)\int_\Omega \left(|\nabla u|^2 + |\triangle u|^2\right)dx - \alpha E(u) \\
&\qquad + \left(1 - \frac{\alpha}{2p+2}\right)\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \\
&\quad = \left(\frac{\alpha}{2} - 1\right)\int_\Omega \left(|\nabla u|^2 + |\triangle u|^2\right)dx - \alpha E(u_0) + \alpha\int_0^t\int_\Omega \left(|u_t|^2 + |\nabla u_t|^2\right)dx\,d\tau \\
&\qquad + \left(1 - \frac{\alpha}{2p+2}\right)\int_\Omega\int_\Omega K(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \\
&\quad \geq \left(\frac{\alpha}{2} - 1\right)\int_\Omega \left(|\nabla u|^2 + |\triangle u|^2\right)dx - \alpha E(u_0)
\end{aligned}$$

$$\geq \left(\frac{\alpha}{2}-1\right)\lambda_1\int_\Omega\left(|\nabla u|^2+|u|^2\right)dx-\alpha E(u_0)$$
$$= m\left[\int_\Omega\left(|\nabla u|^2+|u|^2\right)dx-\eta E(u_0)\right]=mH(t), \tag{3.4}$$

where $\eta=\frac{\alpha}{m}$; $m=(\frac{\alpha}{2}-1)\lambda_1$; $2\leq\alpha\leq 2p+2$; $\lambda_1$ is the first eigenvalue of operator $-\triangle$ under homogeneous Dirichlet boundary conditions.

Due to the conditions (3.1), it follows that

$$H(0)=\int_\Omega\left(|u_0|^2+|\nabla u_0|^2\right)dx-\eta E(u_0)>0. \tag{3.5}$$

Multiplying (3.4) by $e^{-2mt}$, we have

$$e^{-2mt}\frac{d}{dt}H(t)-2me^{-2mt}H(t)=\frac{d}{dt}\left[e^{-2mt}H(t)\right]\geq 0.$$

From the last inequality above and (3.5), we obtain

$$H(t)\geq H(0)e^{2mt}>0. \tag{3.6}$$

From what has been discussed above, we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx>\alpha\int_0^t\int_\Omega\left(|u_t|^2+|\nabla u_t|^2\right)dx\,d\tau. \tag{3.7}$$

Now we define

$$G(t)=\int_0^t\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx\,d\tau. \tag{3.8}$$

Differentiating the identity (3.8) with respect to $t$, we deduce that

$$G'(t)=\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx,$$
$$G''(t)=\frac{d}{dt}\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx\geq 2\alpha\int_0^t\int_\Omega\left(|u_t|^2+|\nabla u_t|^2\right)dx\,d\tau,$$

so we have

$$\begin{aligned}G''(t)G(t)&\geq 2\alpha\int_0^t\int_\Omega\left(|u_t|^2+|\nabla u_t|^2\right)dx\,d\tau\cdot\int_0^t\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx\,d\tau\\&\geq 2\alpha\int_0^t\int_\Omega|u_t|^2\,dx\,d\tau\cdot\int_0^t\int_\Omega|u|^2\,dx\,d\tau\\&\quad+2\alpha\int_0^t\int_\Omega|u_t|^2\,dx\,d\tau\cdot\int_0^t\int_\Omega|\nabla u|^2\,dx\,d\tau\\&\quad+2\alpha\int_0^t\int_\Omega|\nabla u_t|^2\,dx\,d\tau\cdot\int_0^t\int_\Omega|u|^2\,dx\,d\tau\\&\quad+2\alpha\int_0^t\int_\Omega|\nabla u_t|^2\,dx\,d\tau\cdot\int_0^t\int_\Omega|\nabla u|^2\,dx\,d\tau.\end{aligned} \tag{3.9}$$

Using Schwarz's inequality, we get

$$\left(\int_0^t \int_\Omega u u_t \, dx \, d\tau\right)^2 \le \int_0^t \int_\Omega |u_t|^2 \, dx \, d\tau \cdot \int_0^t \int_\Omega |u|^2 \, dx \, d\tau, \tag{3.10}$$

$$\left(\int_0^t \int_\Omega \nabla u \nabla u_t \, dx \, d\tau\right)^2 \le \int_0^t \int_\Omega |\nabla u_t|^2 \, dx \, d\tau \cdot \int_0^t \int_\Omega |\nabla u|^2 \, dx \, d\tau, \tag{3.11}$$

and

$$\begin{aligned} &2\int_0^t \int_\Omega u u_t \, dx \, d\tau \cdot \int_0^t \int_\Omega \nabla u \nabla u_t \, dx \, d\tau \\ &\quad\le 2\left(\int_0^t \int_\Omega |u_t|^2 \, dx \, d\tau\right)^{\frac{1}{2}} \cdot \left(\int_0^t \int_\Omega |u|^2 \, dx \, d\tau\right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\int_0^t \int_\Omega |\nabla u_t|^2 \, dx \, d\tau\right)^{\frac{1}{2}} \cdot \left(\int_0^t \int_\Omega |\nabla u|^2 \, dx \, d\tau\right)^{\frac{1}{2}} \\ &\quad\le \int_0^t \int_\Omega |\nabla u_t|^2 \, dx \, d\tau \cdot \int_0^t \int_\Omega |u|^2 \, dx \, d\tau \\ &\qquad + \int_0^t \int_\Omega |u_t|^2 \, dx \, d\tau \cdot \int_0^t \int_\Omega |\nabla u|^2 \, dx \, d\tau. \end{aligned} \tag{3.12}$$

Inserting (3.10)-(3.12) into (3.9), we find

$$\begin{aligned} G''(t)G(t) &\ge 2\alpha\left(\int_0^t \int_\Omega u u_t \, dx \, d\tau\right)^2 + 2\alpha\left(\int_0^t \int_\Omega \nabla u \nabla u_t \, dx \, d\tau\right)^2 \\ &\quad + 4\alpha \int_0^t \int_\Omega u u_t \, dx \, d\tau \cdot \int_0^t \int_\Omega \nabla u \nabla u_t \, dx \, d\tau \\ &= 2\alpha\left[\int_0^t \int_\Omega (u u_t + \nabla u \nabla u_t) \, dx \, d\tau\right]^2 \\ &= \frac{\alpha}{2}\left(\int_0^t G''(\tau) \, d\tau\right) = \frac{\alpha}{2}\bigl(G'(t) - G'(0)\bigr)^2. \end{aligned} \tag{3.13}$$

Thus, we obtain

$$G''(t)G(t) - \frac{\alpha}{2}\bigl(G'(t) - G'(0)\bigr)^2 \ge 0. \tag{3.14}$$

On the other hand, from (3.6), we know

$$\lim_{t\to\infty} H(t) = +\infty.$$

This implies

$$G'(t) = \int_\Omega \bigl[|u|^2 + |\nabla u|^2\bigr] \, dx \to +\infty, \quad t \to \infty. \tag{3.15}$$

Hence, for $2 < \beta < \alpha$ there exists a $T_\beta$, such that for all $t \ge T_\beta$

$$\alpha\bigl(G'(t) - G'(0)\bigr)^2 \ge \beta G'(t)^2. \tag{3.16}$$

By (3.14) and (3.16), we have

$$G''(t)G(t) - \frac{\beta}{2}G'(t)^2 \geq 0, \quad t \geq T_\beta. \tag{3.17}$$

We consider the function $G(t)^{-q}$ for $0 < q < \frac{\beta}{2}$, we see that

$$\begin{aligned}\bigl(G(t)^{-q}\bigr)'' &= qG(t)^{-q-2}\bigl[(q+1)G'(t)^2 - G''(t)G(t)\bigr] \\ &\leq qG(t)^{-q-2}\left[\frac{2(q+1)}{\beta} - 1\right]G''(t)G(t) < 0, \quad t \geq T_\beta. \end{aligned}\tag{3.18}$$

Since a concave function must always lie below any tangent line, we see that $G(t)^{-q}$ reaches 0 in finite time as $t \to T^-$, where $T > T_\beta$. This means

$$\lim_{t\to T^-} G(t) = +\infty,$$

or

$$\lim_{t\to T^-} \int_0^t \int_\Omega \bigl(|u|^2 + |\nabla u|^2\bigr)\,dx\,d\tau = +\infty. \tag{3.19}$$

Then the desired assertion immediately follows. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed to each part of this work equally and read and approved the final manuscript.

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